Geometry through Trigonometry

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Contents

1 The Right Angled Triangle

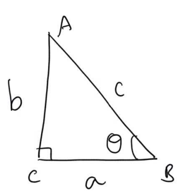

Fig. 1: Right Angled Triangle

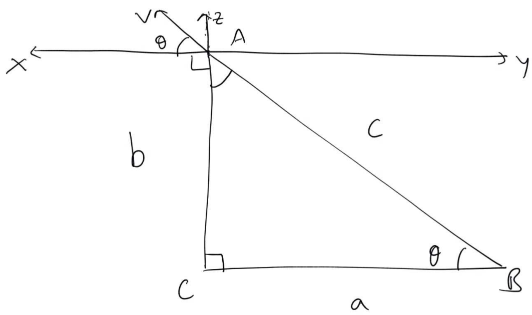

Fig. 2: Sum of angles of a triangle

Problem 1.1. *Show that* $\angle VAZ = 90^{\circ} - \theta$

Problem 1.2. *Show that* $\angle BAC = 90^{\circ} - \theta$*.*

Problem 1.3. *Sum of the angles of a triangle is equal to* 180°

1.1 Budhayana Theorem

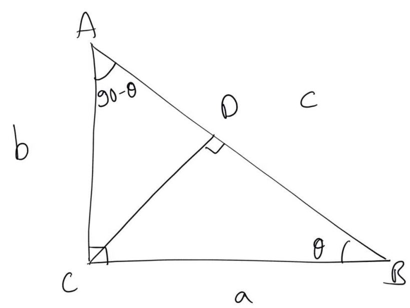

Fig. 3: Budhayana Theorem

Problem 1.4. *Using Fig. 1, show that*

$$\cos\theta = \sin\left(90^{\circ} - \theta\right) \tag{1.1}$$

Problem 1.5. *Using Fig. 3, show that*

$$c = a\cos\theta + b\sin\theta \tag{1.2}$$

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Problem 1.6. *From* (1.2)*, show that*

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.3}$$

Problem 1.7. *Using* (1.2)*, show that*

$$c^2 = a^2 + b^2 \tag{1.4}$$

(1.4) *is known as the Budhayana theorem. It is also known as the Pythagoras theorem.*

2 Medians of a Triangle

2.1 Area of a Triangle

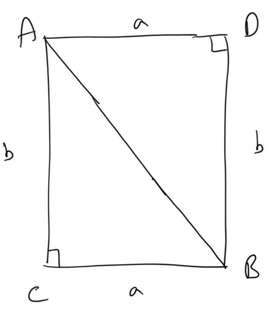

Fig. 4: Area of a Right Triangle

Problem 2.1. *Show that the area of* ΔABC *is* $\frac{ab}{2}$

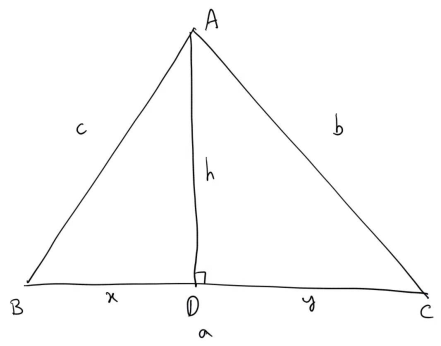

Fig. 5: Area of a Triangle

Problem 2.2. *Show that the area of* ΔABC *in Fig. 5 is* $\frac{1}{2}ah$.

Problem 2.3. *Show that the area of* ΔABC *in Fig. 5 is* $\frac{1}{2}ab\sin C$.

Problem 2.4. *Show that*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{2.1}$$

2.2 Median

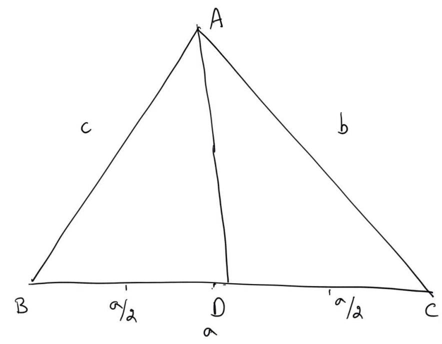

Fig. 6: Median of a Triangle

Problem 2.5. *Show that the median AD in Fig. 6 divides* ΔABC *into triangles ADB and ADC that have equal area.*

Problem 2.6. *BE and CF are the medians in Fig. 7. Show that*

$$ar\,(\Delta BFC) = ar\,(\Delta BEC) \tag{2.2}$$

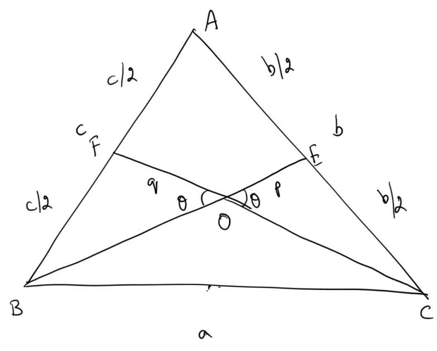

Fig. 7: O is the Intersection of Two Medians

Problem 2.7. *The medians BE and CF in Fig. 7 meet at point O. Show that*

$$\frac{OB}{OE} = \frac{OC}{OF} \tag{2.3}$$

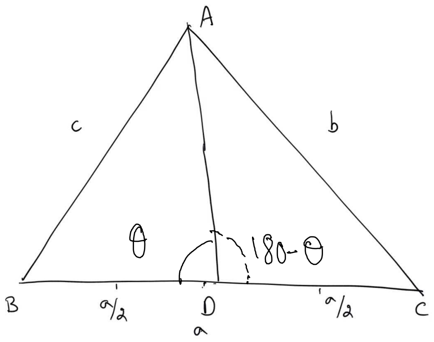

Fig. 8: $\sin\theta = \sin\left(180^{\circ} - \theta\right)$

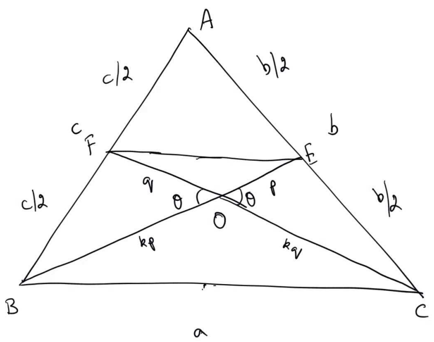

Fig. 9: O divides medians in the ratio $2:1$

Problem 2.8. *In Fig. 9, show that*

$$ar\left(\Delta AFE\right) = \frac{1}{4}ar\left(\Delta ABC\right) \tag{2.4}$$

Problem 2.9. *Using Fig. 9, show that*

$$\frac{OB}{OE} = \frac{OC}{OF} = 2 \tag{2.5}$$

2.3 Similar Triangles

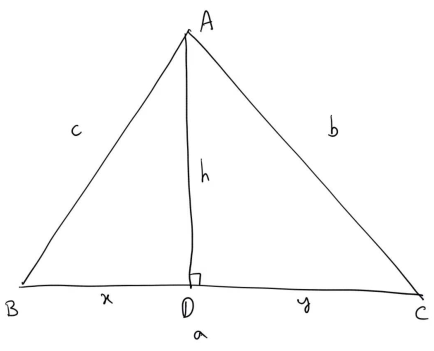

Fig. 10: The cosine formula

Problem 2.10. *In Fig. 10, show that*

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{2.6}$$

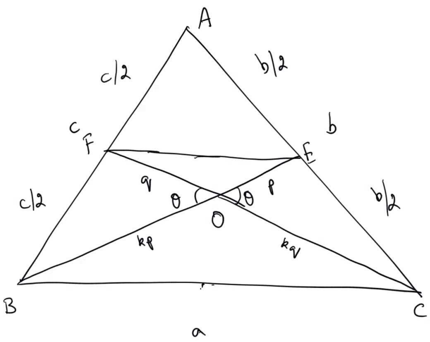

Fig. 11: Similar Triangles

Problem 2.11. *In Fig. 11, show that $EF = \frac{a}{2}$.*

Problem 2.12. *Show that similar triangles have the same angles.*

Problem 2.13. *Show that in Fig. 11, $EF \| BC$.*

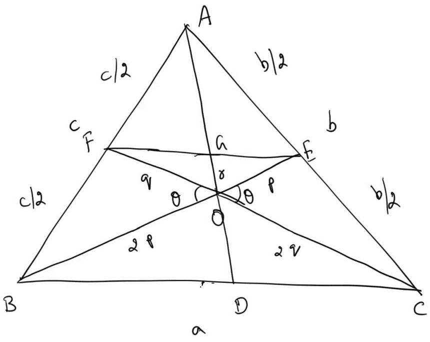

Fig. 12: Similar Triangles

Problem 2.14. *In Fig. 12, the line AO cuts EF at G and is extended to meet the side BC at D. Show that*

$$\frac{OA}{OD} = 2. \tag{2.7}$$

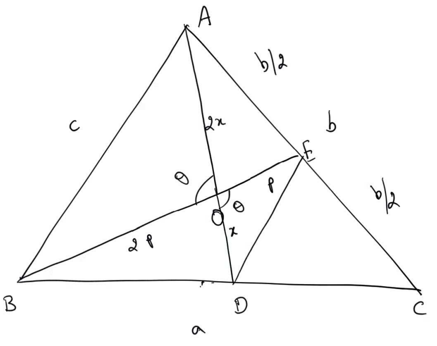

Fig. 13: Medians meet at a point

Problem 2.15. *In Fig. 13,BE is a median of ΔABC and $\frac{OA}{OD} = 2$. Show that AD is also a median.*

3 Angle and Perpendicular Bisectors

3.1 Angle Bisectors

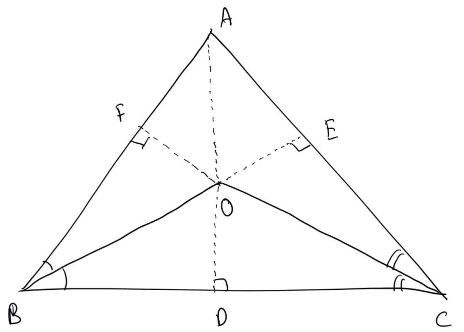

Fig. 14: Angle bisectors meet at a point

Problem 3.1. *Show that $OD = OE = OF$.*

Problem 3.2. *Show that OA is the angle bisector of $\angle A$*

3.2 Congruent Triangles

Problem 3.3. *Show that in Δs ODC and OEC, corresponding sides and angles are equal.*

Problem 3.4. *SSS: Show that if the corresponding sides of three triangles are equal, the triangles are congruent.*

Problem 3.5. *ASA: Show that if two angles and any one side are equal in corresponding triangles, the triangles are congruent.*

Problem 3.6. *SAS: Show that if two sides and the angle between them are equal in corresponding triangles, the triangles are congruent.*

Problem 3.7. *RHS: For two right angled triangles, if the hypotenuse and one of the sides are equal, show that the triangles are congruent.*

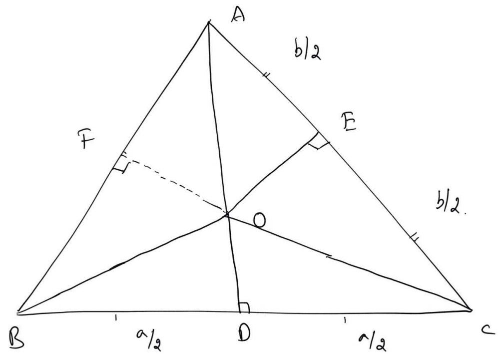

Fig. 15: Perpendicular bisectors meet at a point

Problem 3.8. *In Fig. 15, show that $OA = OB = OC$.*

Problem 3.9. *Show that $AF = BF$.*

3.3 Perpendiculars from Vertex to Opposite Side

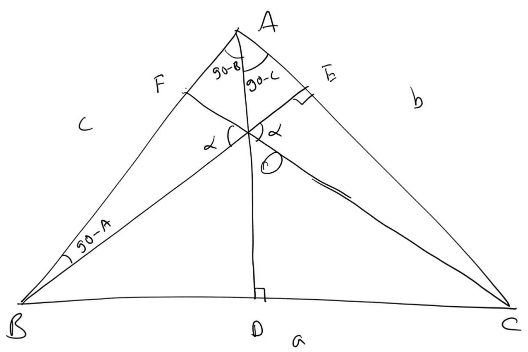

Fig. 16: Perpendiculars from vertex to opposite side meet at a point

In Fig. 16, $AD \perp BC$ and $BE \perp AC$. CF passes through O and meets AB at F.

Problem 3.10. *Show that*

$$OE = c \cos A \cot C \tag{3.1}$$

Problem 3.11. *Show that $\alpha = A$.*

Problem 3.12. *Show that $CF \perp AB$*

4 Circle

4.1 Chord of a Circle

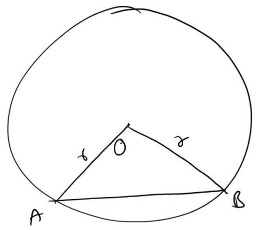

Fig. 17: Circle Definitions

4.2 Chords of a circle

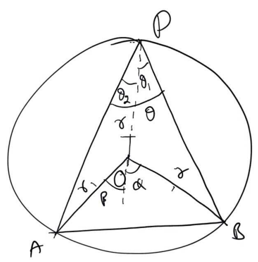

Fig. 18: Angle subtended by chord AB at the centre O is twice the angle subtended at P.

Problem 4.1. *In Fig. 18 Show that $\angle OAB = 2\angle APB$.*

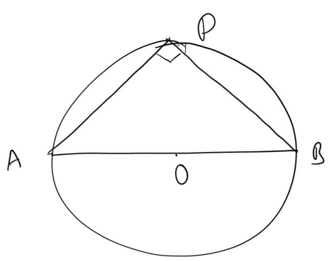

Fig. 19: Diameter of a circle.

Problem 4.2. *In Fig. 19, show that $\angle APB = 90^\circ$.*

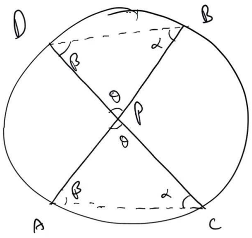

Fig. 20: $PA.PB = PC.PD$

Problem 4.3. *In Fig. 20, show that*

$$\begin{aligned} \angle ABD &= \angle ACD \\ \angle CAB &= \angle CDB \end{aligned} \tag{4.1}$$

Problem 4.4. *In Fig. 20, show that the triangles PAB and PBD are similar*

Problem 4.5. *In Fig. 20, show that*

$$PA.PB = PC.PD \tag{4.2}$$

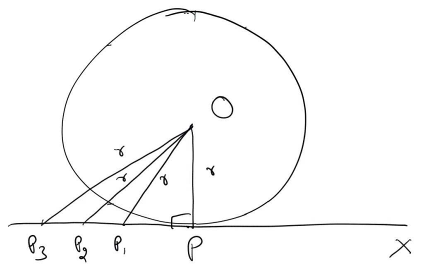

Fig. 21: Tangent to a Circle.

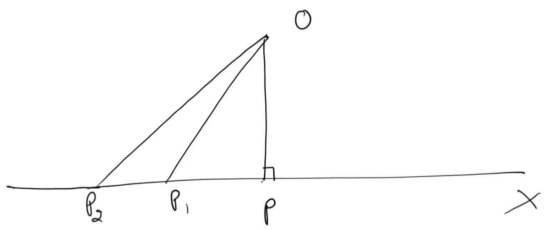

Fig. 22: Shortest distance from O to line PX

Problem 4.6. *OP is the perpendicular to the line PX as shown in the Fig. 22. Show that OP is the shortest distance between the point O and the line PX.*

Problem 4.7. *Show that* $\angle OPX = 90^{\circ}$

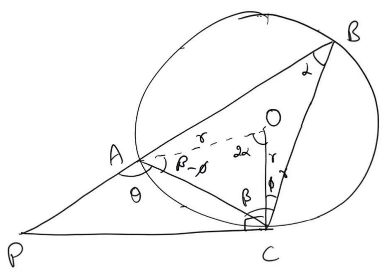

Fig. 23: $PA.PB = PC^2$.

Problem 4.8. *In Fig. 23 show that*

$$\angle PCA = \angle PBC \tag{4.3}$$

O is the centre of the circle and PC is the tangent.

Problem 4.9. *In Fig. 23, show that the triangles PAC and PBC are similar.*

Problem 4.10. *Show that* $PA.PB = PC^2$

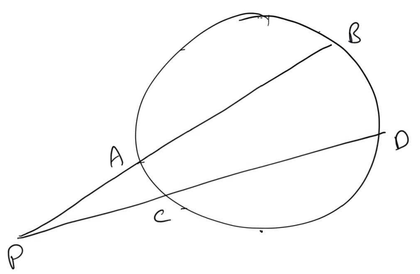

Fig. 24: $PA.PB = PC^2$.

Problem 4.11. *In Fig. 24, show that*

$$PA.PB = PC.PD \tag{4.4}$$

5 Area of a Circle

5.1 The Regular Polygon

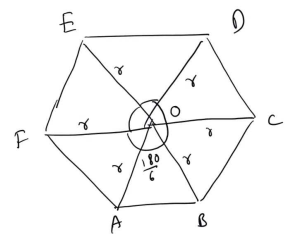

Fig. 25: Polygon Definition

The angle formed by each of the congruent triangles at the centre of a regular polygon of n sides is $\frac{360^{\circ}}{n}$.

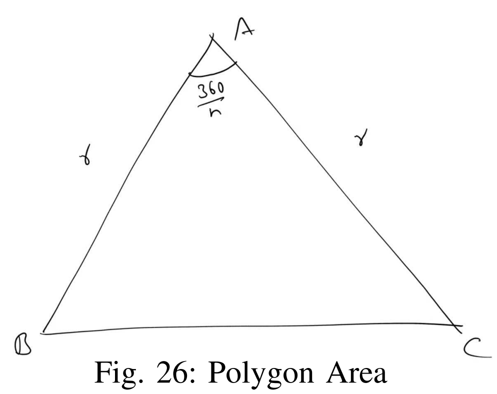

Fig. 26: Polygon Area

Problem 5.1. *Show that the area of a regular polygon is given by*

$$\frac{n}{2} r^2 \sin \frac{360^{\circ}}{n} \tag{5.1}$$

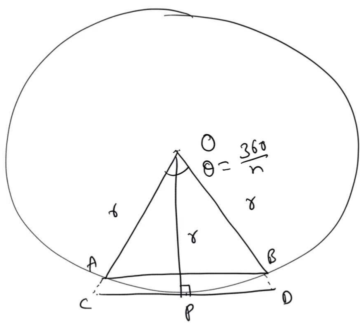

Fig. 27: Circle Area in between Area of Two Polygons

Problem 5.2. *Using Fig. 27, show that*

$$\frac{n}{2}r^2 \sin\frac{360^\circ}{n} < \text{ area of circle } < nr^2 \tan\frac{180^\circ}{n} \tag{5.2}$$

The portion of the circle visible in Fig. 27 is defined to be a sector of the circle.

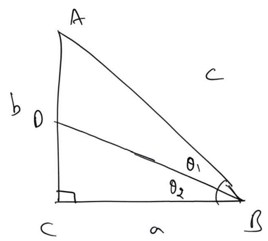

Fig. 28: $\sin 2\theta = 2\sin\theta\cos\theta$

Problem 5.3. *Using Fig. 28, show that*

$$\sin\theta_1 = \sin(\theta_1+\theta_2)\cos\theta_2 - \cos(\theta_1+\theta_2)\sin\theta_2 \tag{5.3}$$

Problem 5.4. *Prove the following identities*

1) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$
2) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

Problem 5.5. *Using* (5.3) *and* (2)*, show that*

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \tag{5.4}$$

$$\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2 \tag{5.5}$$

Problem 5.6. *Show that*

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{5.6}$$

Problem 5.7. *Show that*

$$\cos^2\frac{180^\circ}{n} < \frac{\text{area of circle}}{nr^2\tan\frac{180^\circ}{n}} < 1 \tag{5.7}$$

Problem 5.8. *Show that if*

$$\theta_1 < \theta_2, \sin\theta_1 < \sin\theta_2. \tag{5.8}$$

Problem 5.9. *Show that if*

$$\theta_1 < \theta_2, \cos\theta_1 > \cos\theta_2. \tag{5.9}$$

Problem 5.10. *Show that*

$$\sin 0^\circ = 0 \tag{5.10}$$

Problem 5.11. *Show that*

$$\cos 0^\circ = 1 \tag{5.11}$$

Problem 5.12. *Show that for large values of n*

$$\cos^2\frac{180^\circ}{n} = 1 \tag{5.12}$$

Problem 5.13. *Show that*

$$\text{area of circle } = r^2\lim_{n\to\infty} n\tan\frac{180^\circ}{n} \tag{5.13}$$

Problem 5.14. *Show that the circumference of a circle is* $2\pi r$.

Problem 5.15. *Show that the area of a sector with angle* θ *in radians is* $\frac{1}{2}r^2\theta$.

Problem 5.16. *Show that*

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \tag{5.14}$$